

Taylor's dissipation surrogate and its associated anomaly

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Abstract

It is shown that, for stationary isotropic turbulence, Taylor's well known dissipation surrogate Du'^3/L can be derived directly from the Karman-Howarth equation and is in fact a surrogate for inertial transfer, which becomes equal to the dissipation, as the Reynolds number tends to infinity. The expression found for the dissipation rate ε is

$$\varepsilon = \frac{A_3 u'^3}{L} \left[1 + \frac{1}{R_L} \frac{A_2}{A_3} \right],$$

where the coefficients A_2 and A_3 depend on the second- and third-order structure functions respectively and $R_L = u'L/\nu$ is the Reynolds number based on the integral length scale L . Further, consideration of the spectral energy transfer processes shows that the dissipation rate is entirely determined by the energy injection rate. The role of the viscosity is merely to determine the way in which the turbulent system adapts to increasing injection rates, both by increasing the volume of k -space and by changing the shape of the energy spectrum. The quantity u'^3/L is a measure of the inertial transfer rate and only becomes equal to the dissipation when the Reynolds number is large enough to permit scale-invariance. It is noted that similar conclusions hold for shear flows, such as Poiseuille flow, both laminar and turbulent, where the analogue of energy injection is the rate of doing work expressed in terms of the pressure gradient and the mean velocity.

1 Introduction

The idea that the rate of kinetic energy dissipation per unit mass of fluid turbulence has something anomalous about it, is widespread: see [1]-[4] and references therein. It arises from a result established in 1935 by Taylor [5], who showed, essentially by dimensional arguments, that the dissipation rate could be written as proportional to u'^3/l where u' is the rms velocity of the fluid and l is some length scale of the system. Later, Batchelor [6] discussed this idea further in terms of the decay of kinetic energy (of turbulent motion) and offered two interpretations of it. The first of these was to see it as the decay of an amount of energy u'^2 in a time l/u' . His second interpretation was to regard it as the effect of an eddy viscosity ul acting on a shear of order u/l to ‘produce a “dissipation” (*sic*) of energy from the energy-containing eddies to smaller eddies’. It seems likely from the context, and his use of quotation marks, that Batchelor saw this kind of expression as an approximation to inertial transfer which would be equal to the *actual* dissipation, under conditions of local statistical equilibrium. He also made use of the integral length scale, while noting that it was not as directly representative of the energy containing eddies as it might be.

Essentially this is a pragmatic choice and in recent years the integral length scale L has been generally employed. Taylor’s expression is normally written as

$$\varepsilon = Du'^3/L, \quad (1)$$

where D tends to a constant¹ with increasing Reynolds number [6]-[12], and is widely used as a surrogate expression for the dissipation.

Taylor’s analysis suggested that D would depend on the geometrical nature of the boundary conditions; while Batchelor pointed out that it might depend on the time of decay, the initial conditions of the turbulence, and the choice made for l . This view has received recent theoretical support [13],[14].

Now let us turn our attention to the so-called *dissipation anomaly*. Given that dissipation is due to viscosity and that the dissipation rate is formally defined in terms of the coefficient of viscosity, thus:

$$\varepsilon = \left\langle \frac{\nu}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \right\rangle, \quad (2)$$

the perceived anomaly takes one of two related forms:

1. The fact that the dissipation rate, as given by (1) and verified by experiment, is found to be independent of the fluid viscosity [7]-[12], providing that the Reynolds number is sufficiently large.
2. The existence of finite dissipation in the limit of vanishingly small viscosity (or the limit of infinite Reynolds number) [1]-[3].

Noting that (1) has only been derived from dimensional considerations, and analysed qualitatively in terms of the kinetic energy u'^2 and the eddy turnover time L/u' , there is a need for a more theoretical analysis. From our present point of view, the recent work of Doering and Foias [15] is of interest. For the case of forced turbulence, they have

¹Some workers in the field use C_ε for this constant.

established both upper and lower bounds on the dissipation rate. We shall return to this work at appropriate points in our own analysis.

In this paper we mainly confine our attention to stationary, isotropic turbulence. We examine the exact relationships expressing conservation of energy, first in real space, and then in wavenumber space. We show that Taylor's expression follows quite naturally from the Karman-Howarth equation, as the Reynolds number increases. In the process, we find that it is a surrogate for the inertial transfer, and not for the dissipation rate, as such. Then, by considering the equivalent k -space relation, we verify that the rate of doing work by body forces is the controlling quantity which determines the dissipation rate in this kind of turbulence; and by, fairly obvious extension, in all fluid flows.

2 Taylor's surrogate and the Karman-Howarth equation

The Karman-Howarth equation may be written in terms of structure functions as [16]:

$$-\frac{2}{3}\varepsilon - \frac{1}{2}\frac{\partial S_2}{\partial t} = \frac{1}{6r^4}\frac{\partial}{\partial r}(r^4 S_3) - \frac{\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial S_2}{\partial r}\right), \quad (3)$$

where the structure function of order n is given by

$$S_n = \langle (u(x+r) - u(x))^n \rangle. \quad (4)$$

In this section we first consider the stationary case; then the relationship of the work we present here to other recent work on the subject; and finally the extension of our analysis to freely decaying turbulence.

2.1 Stationary turbulence

For the case of stationary turbulence, we may set the time-derivative equal to zero, and re-arrange the Karman-Howarth equation to obtain an expression for the dissipation, thus:

$$\varepsilon = -\frac{1}{4r^4}\frac{\partial}{\partial r}(r^4 S_3) + \frac{3\nu}{2r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial S_2}{\partial r}\right). \quad (5)$$

Now make the *change of variables*,

$$S_n = V^n f_n(x), \quad \text{with} \quad x = \frac{r}{b}, \quad (6)$$

where V is some constant velocity scale and b is any length scale. We should note that this step involves no approximations or non-trivial assumptions. It merely introduces the f_n as dimensionless forms of the structure functions. As they are dimensionless, their dependence on r must be scaled by some length, here denoted by b . In Section 2.3 we will discuss the introduction of *self-similar* and *similarity solutions* when we extend the present analysis to time-dependent (i.e. freely decaying) turbulence.

With these substitutions, equation (5) becomes:

$$\varepsilon = \frac{A_3 V^3}{b} + \frac{A_2 \nu V^2}{b^2}, \quad (7)$$

where the coefficients A_3 and A_2 are given by

$$A_3 = -\frac{1}{4x^4} \frac{\partial}{\partial x} \left[x^4 f_3(x) \right], \quad (8)$$

and

$$A_2 = \frac{3}{2x^4} \frac{\partial}{\partial x} \left[x^4 \frac{\partial f_2}{\partial x} \right]. \quad (9)$$

Taking out common factors, it is readily seen that the expression for the dissipation becomes

$$\varepsilon = \frac{A_3 V^3}{b} \left[1 + \frac{1}{R_b} \frac{A_2}{A_3} \right], \quad (10)$$

where the Reynolds number is given by $R_b = Vb/\nu$. We note that this equation is still just the Karman-Howarth equation: no approximation has been made.

Now we may identify the prefactor on the right hand side of (10) as being the same as the Taylor dissipation surrogate (1); providing we make the choices $V = u'$, the root-mean-square velocity, and $b = L$, the integral scale. Then we obtain the general relation

$$\varepsilon = \frac{A_3 u'^3}{L} \left[1 + \frac{1}{R_L} \frac{A_2}{A_3} \right], \quad (11)$$

but now the Reynolds number is given by $R_L = u'L/\nu$. Note that the f_n are determined by this choice and hence also the coefficients A_3 and A_2 .

Clearly, as the Reynolds number goes to infinity, this expression reduces to Taylor's surrogate for the dissipation (1), provided that A_3 becomes a constant. However, as it represents the inertial-transfer term, we should really describe it as Taylor's surrogate for inertial transfer.

2.2 Comparison with other work

It is also of interest to compare equation (7), which is an intermediate stage in our calculation, to the result for an upper bound on the dissipation as given by Doering and Foias [15]. This is featured in their abstract as

$$\varepsilon \leq c_1 \nu \frac{u'^2}{l^2} + c_2 \frac{u'^3}{l},$$

and corresponds to their equation (40). Here the coefficients c_1 and c_2 depend on the shape of the forcing function, while l is its longest length-scale. Obviously this is quite different from our own result, where the corresponding parameters depend on the fluid turbulence and not on the forcing, and we have an equality, rather than an inequality. Nevertheless, the general similarity of the two results is worthy of notice.

After some further manipulations, these authors introduce a function β (the same as our D in (1)) such that

$$\beta \equiv \frac{\varepsilon l}{U^3} \leq \left(\frac{a}{Re} + b \right), \quad (12)$$

where the symbols are all as in [15]. They then use the identity $Re = \beta R_\lambda^2$ to substitute for Re , solve the resulting quadratic equation, and obtain

$$\beta \leq \frac{b}{2} \left[1 + \sqrt{1 + \frac{4a}{b^2 R_\lambda^2}} \right], \quad (13)$$

in terms of the Taylor-Reynolds number R_λ .

An interesting development is that Donzis, Sreenivasan and Yeung [11] have taken this upper bound as an equality, which they write as

$$\beta = A \left(1 + \sqrt{1 + (B/R_\lambda)^2} \right). \quad (14)$$

They fit this curve to results obtained from a numerical simulation and obtain an impressively close fit, with $A \sim 0.2$ and $B \sim 92$, leading to an asymptotic value of $\beta = 0.4$.

Evidently our equation (11), being readily reduced to the same form as (12), may be further reduced to the same form as (14), with

$$A = A_3/2 \quad \text{and} \quad B = 2A^{1/2}/A_3. \quad (15)$$

Thus the implication is that it also agrees well with the results from simulation. We are currently doing our own numerical calculations; but, in the meantime, it is reassuring to know this.

2.3 Extension to freely decaying turbulence

We now consider whether our results can also be applied to freely decaying isotropic turbulence. The neglect of the time-derivative term in (3) is usual, even for freely decaying turbulence, provided that the Reynolds number is large enough and one restricts attention to the inertial range. For instance, this step is required in order to derive the well-known ‘4/5’ law and is known as *local stationarity*.

Let us consider the effect on (6) of choosing V and b to be the Kolmogorov velocity and length scales, as given by:

$$v = (\nu\varepsilon)^{1/4} \quad \text{and} \quad \eta = (\nu^3/\varepsilon)^{1/4}; \quad (16)$$

see equation (21.4) in [17] or equations (2.131), (2.132) in [18]. For $n = 2$, substitute (16) into (6) for V and b , respectively. We obtain

$$S_2(r) = \nu^{1/2} \varepsilon^{1/2} f_2(r/\eta). \quad (17)$$

If we are to make any further progress, then we must assume a (specific) *self-similar* form for f_2 . That is, we assume:

$$f_2(r/\eta) = \frac{1}{\eta^{2/3}} f_2(r), \quad (18)$$

from which the second-order structure function becomes:

$$S_2(r) = \varepsilon^{2/3} f_2(r); \quad f_2(r) = r^{2/3}. \quad (19)$$

Similarly, we can show that the third-order structure function is of the form

$$S_3(r) = \varepsilon f_3(r); \quad f_3(r) = r. \quad (20)$$

It should be noted that the forms in (16) omit constants of order unity and this is reflected in these results. Also, we do not consider the question of non-canonical exponents or intermittency corrections as these matters have been addressed elsewhere [19].

Evidently our analysis is consistent with the K41 picture and should apply for the case of local stationarity. Let us now consider a more general situation, where we assume that

the decaying turbulence is self-preserving. Now we replace (6) by the time-dependent form:

$$S_n(r, t) = u'^n(t_e) f_n(x), \quad (21)$$

where

$$x = r/L(t). \quad (22)$$

Note that our choice of an initial time for a *similarity solution* requires some care. We have taken it to be some $t = t_e$, when the turbulence is said to have evolved from arbitrary initial conditions. If we assume that the turbulence is self-preserving after $t = t_e$, then there is no explicit time-dependence in the dimensionless structure functions, and equation (11) applies in the present case as well.

In practice self-preservation is only likely to be approximately correct, and we can take account of the residual time-dependence as a perturbation of the similarity solution, writing equation (21) as

$$S_n(r, t) = u'^n(t_e) f_n(x, \tau), \quad (23)$$

where now

$$\tau = t/T; \quad T = L(t)/u'(t_e). \quad (24)$$

In these circumstances it is easily shown that equation (11) can be generalised to the form:

$$\varepsilon = \frac{(A_3 - B_2)u'^3}{L} \left[1 + \frac{1}{R_L} \frac{A_2}{A_3 - B_2} \right], \quad (25)$$

where the new coefficient B_2 is given by

$$B_2 = \frac{3}{4} \frac{\partial f_2}{\partial \tau}, \quad (26)$$

for small values of the time derivative.

In all, this suggests that equation (11), although derived for stationary turbulence, may apply quite well to decaying turbulence.

3 The spectral picture

In order to examine these ideas further, we study the spectral energy balance in wavenumber space, and consider the effect of increasing the Reynolds number, ultimately taking the limit of infinite Reynolds numbers. The three energy transport processes, viz., injection, dissipation and inertial transfer will now be considered in turn.

We begin with the energy balance equation, in its well known form,

$$\left(\frac{\partial}{\partial t} + 2\nu k^2 \right) E(k, t) = T(k, t) + W(k), \quad (27)$$

where the energy transfer spectrum $T(k, t)$ is given by:

$$T(k, t) = 2\pi k^2 M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \{ C_{\beta\gamma\alpha}(\mathbf{j}, \mathbf{k} - \mathbf{j}, -\mathbf{k}, t) - C_{\beta\gamma\alpha}(-\mathbf{j}, -\mathbf{k} + \mathbf{j}, \mathbf{k}, t) \}, \quad (28)$$

and $C_{\alpha\beta\gamma}(\mathbf{k}, \mathbf{j}, \mathbf{k} - \mathbf{j})$ is the three-velocity correlation. We may write $T(k, t)$ as

$$T(k, t) = \int_0^\infty S(k, j; t) dj, \quad (29)$$

where S depends on the triple moment: its form can be deduced from (28). It can be shown that S is antisymmetric under the interchange $k \rightleftharpoons j$:

$$S(k, j; t) = -S(j, k; t). \quad (30)$$

Hence

$$\int_0^\infty T(k, t) dk = \int_0^\infty dk \int_0^\infty dj S(k, j; t) = 0, \quad (31)$$

is an exact symmetry which expresses conservation of energy.

In order to study the stationary case, we have added an input spectrum $W(k)$ (if one wishes, this can be related to the covariance of the random stirring forces [18]). We also introduce ε_W as the rate at which the stirring forces do work on the turbulent fluid:

$$\varepsilon_W = \int_0^\infty W(k) dk. \quad (32)$$

The dissipation rate ε_D in k -space is defined by $\varepsilon_D = -dE/dt$ for freely decaying turbulence. As is well known, we can obtain an expression which is also valid for the stationary case, in the usual way, by temporarily setting $W(k) = 0$, integrating (27) over wavenumber, and rearranging, such that the energy balance becomes:

$$\varepsilon_D = -\frac{dE}{dt} = \int_0^\infty 2\nu k^2 E(k, t) dk, \quad (33)$$

where we have also invoked equation (31). The region in k -space where the dissipation mainly occurs is characterised by the Kolmogorov dissipation wavenumber:

$$k_D = (\varepsilon_D/\nu^3)^{1/4}. \quad (34)$$

Then, restoring the injection spectrum, for the stationary case we have $dE(k, t)/dt = 0$, and the energy balance becomes:

$$T(k) + W(k) - 2\nu k^2 E(k) = 0. \quad (35)$$

Integrating both sides with respect to wavenumber, we have:

$$\int_0^\infty W(k) dk - \int_0^\infty 2\nu k^2 E(k) dk = 0; \quad \text{or:} \quad \varepsilon_W = \varepsilon_D, \quad (36)$$

as a rigorous consequence of stationarity. In other words, for a stationary flow, the dissipation is governed by the rate at which we do work on the fluid in order to produce a required fluid motion. We shall enlarge on this point presently.

3.1 Inertial transfer and scale-invariance

We now consider inertial transfer of energy in k -space. The energy flux is introduced when we integrate each term of (27) with respect to wavenumber, from zero up to some wavenumber κ . Reverting for the moment to the general (i.e. non-stationary) case, we obtain:

$$\frac{d}{dt} \int_0^\kappa dk E(k, t) = - \int_\kappa^\infty dk \int_0^\kappa dj S(k, j; t) - 2\nu \int_0^\kappa dk k^2 E(k, t), \quad (37)$$

where we have used the antisymmetry of S , and made some rearrangements. In this form the effect of the transfer term is readily interpreted as the net flux of energy from

wavenumbers less than κ to those greater than κ , at any time t [6]. Denoting this flux by $\Pi(\kappa)$; and, in order to avoid the ambiguity associated with the scale-invariance paradox, making an exact decomposition of the transfer spectrum into filtered-partitioned forms $T^{+-}(k|\kappa)$ and $T^{-+}(k|\kappa)$ [20], we have

$$\Pi(\kappa) = \int_{\kappa}^{\infty} dk T^{+-}(k|\kappa) = - \int_0^{\kappa} dk T^{-+}(k|\kappa), \quad (38)$$

where we have now assumed stationarity and dropped the time dependence. (Note that the decomposition is completed by $T^{--}(k|\kappa)$ and $T^{++}(k|\kappa)$, which are separately conservative on the intervals $[0, \kappa]$ and $[\kappa, \infty]$, respectively [20].) The maximum value of the energy flux is $\Pi_{max}(\kappa)$, where $T^{+-}(k|\kappa) = T^{-+}(k|\kappa) = 0$.

When setting up stationary, isotropic turbulence by means of an arbitrary choice of stirring forces, it is usual to try to reproduce the characteristic features of the classic turbulent shear flows. In terms of the energy spectrum, these may be seen as the energy-containing range (low wavenumbers), the inertial range (intermediate wavenumbers) and the dissipation range (large wavenumbers). It has been known since the late 1930s [6] that the energy-containing and dissipation ranges become more widely separated as the Reynolds number is increased; with the Kolmogorov dissipation wavenumber k_D , as given by (34), providing a reliable measure of this process. In practice, what this means is that we should choose the injection spectrum $W(k)$ to have a somewhat peaked form at low values of wavenumber, when compared to k_D .

We may formalize this picture as follows. At sufficiently large R_λ , the energy-containing and dissipation ranges become separated by the inertial range of wavenumbers, thus:

$$k_{bot} \leq \kappa \leq k_{top}, \quad (39)$$

where κ now stands for any wavenumber in the inertial range. In this case, the injection and dissipation spectra satisfy approximate relationships as follows:

$$\int_0^{k_{bot}} dk W(k) \simeq \varepsilon_W; \quad \text{and} \quad \int_{k_{top}}^{\infty} dk 2\nu k^2 E(k) \simeq \varepsilon_D. \quad (40)$$

In this range of wavenumbers the maximum energy flux should be approximately constant and we will find it helpful to introduce a specific symbol for this quantity, thus:

$$\Pi_{max} = \varepsilon_T. \quad (41)$$

For stationarity, we must have the overall energy balance:

$$\varepsilon_W = \varepsilon_T = \varepsilon_D. \quad (42)$$

We note that these three different physical processes are normally denoted by the single symbol ε , this being justified by their all being numerically equal. In our view, it is necessary to draw a distinction between them in order to avoid confusion.

It should now be apparent that Taylor's expression, as given by (1), should be written as $\varepsilon_T = Du^3/L$; which also explains the observed dependence on Reynolds number (see Fig. 1 in [8]). In other words, (1) becomes equal to the dissipation when (42) is satisfied.

3.2 The limit of infinite Reynolds numbers

The separation of the energy-containing and dissipation ranges by a scale-invariant inertial range has an interesting consequence. It allows us to obtain separate low- k and high- k balance equations; by first integrating from zero up to k_{bot} , and then from infinity down to k_{top} . First, we have

$$\int_0^{k_{bot}} dk \int_{k_{bot}}^{\infty} dj S(k, j) + \int_0^{k_{bot}} W(k) dk = 0. \quad (43)$$

That is, energy supplied directly by the input term to modes with $k \leq k_{bot}$ is transferred by the nonlinearity to modes with $j \geq k_{bot}$. Thus $T(k)$ behaves like a *dissipation* and absorbs energy. Second,

$$\int_{k_{top}}^{\infty} dk \int_0^{k_{top}} dj S(k, j) - \int_{k_{top}}^{\infty} 2\nu k^2 E(k) dk = 0. \quad (44)$$

That is, the nonlinearity transfers energy from modes with $j \leq k_{top}$ to modes with $k \geq k_{top}$, where it is dissipated into heat. In this range of wavenumbers $T(k)$ behaves like a *source* and emits energy which is then dissipated by viscosity.

Now we reach the crux of our argument. We examine the limit of infinite Reynolds number, as follows. We keep the injection spectrum $W(k)$ fixed (and hence also $\varepsilon_W = \varepsilon_T = \varepsilon_D$ are held constant), and allow the viscosity to tend to zero. As a result (originally pointed out by Batchelor in 1953 [6]), the sink of energy is displaced to $k = \infty$, in the limit of infinite Reynolds number. We can deduce that this is so, either from the form of the Kolmogorov dissipation wavenumber or the form of the dissipation term; or even from a consideration of the local Reynolds number for mode k [6].

However, we cannot emphasise too strongly that this has nothing to do with the Euler equation. The Euler equation can indeed be obtained from the Navier-Stokes equation by setting the viscosity equal to zero. The result is the equation of motion for an inviscid, zero-dissipation fluid. Here, in contrast, we take the *limit* of zero viscosity, while keeping the dissipation constant.

This operation was later formalized by Edwards [21] (or see Section 6.2.7 in [18]), who argued that the dissipation at infinity could be represented by a Dirac delta function, thus:

$$\int_0^{\infty} dk \lim_{\nu \rightarrow 0} 2\nu k^2 E(k) \Big|_{\varepsilon_D = \text{constant}} = \int_0^{\infty} dk \varepsilon_D \delta(k - \infty) = \varepsilon_D. \quad (45)$$

Similarly, with an infinite extent of wavenumber space, any injection spectrum of finite extent can be scaled back to the origin, so that we have also:

$$\int_0^{\infty} dk \lim_{\nu \rightarrow 0} W(k) \Big|_{\varepsilon_W = \text{constant}} = \int_0^{\infty} dk \varepsilon_W \delta(k) = \varepsilon_W. \quad (46)$$

Thus, in the limit of infinite Reynolds numbers, equation (35) for the spectral energy balance may be written as:

$$-T(k) = \varepsilon_W \delta(k) - \varepsilon_D \delta(k - \infty). \quad (47)$$

Note that these forms satisfy all the relevant relationships given above as equations (39) - (44) and, although this may seem a rather extreme procedure, it is in fact nothing more than a different mathematical representation of scale invariance in the inertial range.

The general forms of our arguments given here in spectral space are not unlike those of Davidson (see Section 3.2.2 in [22]), which are presented for physical or scale space. A particular point of interest is his example of a sudden expansion in a pipe, where the ‘head loss’ is determined by continuity and the momentum theorem (plus, it must be said, an approximation based on the existence of stagnation points) but not apparently on the fluid viscosity.

Also, it is of interest to note that our views expressed here seem to be quite close to those of Doering and Foias [15], despite the very different form of the analysis. Essentially both treatments give a dominant role to the production of the turbulence and, for instance, our equation (36) is essentially just equation (17) of reference [15].

4 Conclusion

From this simple analysis, it is apparent that there are really no grounds for viewing the lack of dependence of the dissipation on the viscosity as an anomaly. Conservation of energy ensures that the dissipation is just equal to the injection rate. The role of the viscosity is to control the energy occupation of wavenumber space, along with the shape of the spectrum. Noting that Taylor’s expression for the dissipation, as given by equation (1), is really just the inertial transfer rate, we can appreciate that it will depend on Reynolds number, until the latter is large enough for the energy-containing and dissipation ranges of wavenumber to be adequately separated, to the point where inertial transfer is well-defined, and equal to the actual dissipation rate. Also, for the limit of infinite Reynolds number, it can be seen that the symmetry-breaking effect of the viscosity is still present in the form of the delta functions, as in equation (31).

We can also attempt to say something more general about shear flows. If we consider classical shear flows, then laminar Poiseuille flow presents one of the few cases where the Navier-Stokes equation may be solved exactly. This is because the nonlinear term vanishes identically, and the problem resolves itself into a balance between viscous and pressure forces. Here the dissipation rate can be worked out exactly, and is found to be independent of viscosity. It is, as it must be, equal to the injection rate; which can also be worked out exactly, in terms of the rate at which the pressure force moves its point of application. In this case, the gradient of the velocity depends inversely on the viscosity, and becomes steeper in order to maintain the dissipation rate as the viscosity is decreased at constant axial pressure gradient. This is, of course, analogous to the behaviour of the Kolmogorov length scale under increasing Reynolds number at constant injection rate.

Extension of the analysis to turbulent Poiseuille flow requires a little more work, but those nonlinear terms which do not vanish identically, do vanish when integrated over the system volume; and the analysis for the laminar case applies here as well.

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